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AN ANALYSIS OF SHALLOW SPHEROIDAL SHELLS BY A SEMI-INVERSE CONTOUR METHOD

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Abstract—This paper presents a linear analysis of a shallow prolate spheroidal shell with a planar elliptical boundary. The shell is subjected to a uniform load, q, and clamped along the boundary. The theory used in this paper is characterized by the well known Mushtari-Donnell-Vlasov equations which consist of a compatibility equation and an equilibrium equation where the normal displacement, w, and a stress function, ϕ, are the dependent variables.

The method employed for the solution of this problem is developed in three major stages. The first stage involves the determination of w, under the assumption that the contours of w be ellipses concentric to the boundary. The second stage is devoted to the determination of a stress function ϕ which, together with w, satisfies the MDV compatibility equation exactly. The third stage of the development is concerned with the computation of a loading, q^*, which, together with w and ϕ, satisfies the equilibrium equation exactly and which is nearly equal to the desired uniform loading q.

NOTATION

A_1, A_2	Lame' parameters
a	Major axis of the complete spheroid
B_{jm}	Constants
b	Minor axis of the complete spheroid
C_1, C_2, C_3	Constants of integration
c_1, c_2, c_3	$C_1/w_0, C_2/w_0, C_3/w_0$ respectively
E	Modulus of elasticity
$E(x), F(x)$	Bessel-type functions
F_{jm}	Constants
H_0	Height of shallow spheroid
$H(i, n, j, m)$	Transformation coefficients
h	Thickness
k	b/a
L_0	Major axis of shallow spheroid
l_0	Minor axis of shallow spheroid
p	$w_0{}^P/Q$
Q, Q^*	Uniform and nonuniform distributions respectively
Q_{nj}, Q^*_{nj}	Expansion coefficients for Q, Q^* respectively
q, q^*	Uniform and nonuniform loadings respectively
R_1, R_2	Radii of curvature
$s(k)$	Shape function
t	Angular coordinate
u_1, u_2	Tangential displacements
w	Normal displacement
$w_0, w_0{}^P$	Maximum displacements for the shallow spheroid and an elliptical plate, respectively
$\bar{w}$	w/w_0
$w^{(i)}$	Test solutions for w
$w_n{}^{(i)}, w_{nj}$	Expansion coefficients for $w^{(i)}$ and w respectively
X_1, X_2, X_3	Cartesian coordinates
x, x_0	Contour line coordinate and its value on the boundary, respectively
α_1, α_2	Principal coordinates
δ	Constant
θ	Angular coordinate
$\lambda, \bar{\lambda}$	Contour line coordinates
λ_0	Value of λ on the boundary
μ, μ_0, μ_1	Constants
ν	Poisson's ratio
ξ_1, ξ_2	Principal coordinates
ϕ, Φ	Stress functions
Φ_{nj}	Expansion coefficients for Φ
ψ	Error function

1. INTRODUCTION

A generalization of the semi-inverse method used in classical elasticity theory is employed in this paper to analyze a thin shallow prolate spheroidal shell with a planar elliptical boundary (see Fig. 1). The shell is subjected to a uniform load and clamped along the boundary.

Because of the elliptical shape of the boundary curve of the shell, the concept of axial (or rotational) symmetry cannot be utilized to simplify the analysis. Further, a coordinate system involving the lines of principal curvature will not include the boundary curve as a member of the family of coordinate lines—thus making it very difficult to express the boundary conditions.

The bending analysis of a shallow shell of elliptical planform was apparently first discussed by Surkin[1] in an approximate analysis of the post buckling behavior of prolate spheroidal shells under uniform pressure. An improved Rayleigh–Ritz formulation of the problem was presented by Hyman[2] in terms of a coordinate system in which one family of the new coordinate curves coincides with the edge of the shallow cap, thus greatly simplifying the boundary conditions. This family of curves, constructed by intersecting the middle surface with planes parallel to the plane containing the base ellipse, will henceforth be referred to as the "constant λ (or $\bar{\lambda}$) lines". In that paper, Hyman assumed that both the normal displacement w and the in-plane displacements of shallow spheroidal shells which are nearly circular in planform are dependent upon λ only.

These constant λ lines also appeared in the works by Nash[3] in the large deflection analysis of elliptical plates, by Mazumdar[4–7] in the analysis of plates and membranes, by Broome[8, 9], Jones[10], and by Jones and Mazumdar[11] in the analysis of the shallow spheroidal shell which is under consideration here.

In Jones' analysis, the governing equations are in the form of two integro-differential equilibrium equations and one differential continuity equation, with the unknowns being the vertical displacement w, a stress function ϕ, and a function expressing the contour curves of w. By assuming that the contour curves coincide with the constant λ lines, i.e. $w = w(\lambda)$, Jones and Mazumdar show that the two equilibrium equations are identical, thus, reducing the problem to two equations with two unknowns. Exact solutions to these two equations, were obtained by further assuming that $\phi = \phi(\lambda)$.

In this paper we present a semi-inverse contour method developed by Broome[9] for solving the problem of the shallow spheroid subjected to a uniform loading, q. The shell theory used in this paper is characterized by the well-known Mushtari–Donnell–Vlasov (MDV) equations which consist of a compatibility equation and an equilibrium equation where the normal displacement, w, and a stress function, ϕ, are the dependent variables. The semi-inverse contour method as utilized herein involves the selection of w and the determination of ϕ such that the compatibility equation and the equilibrium equation are satisfied exactly for a loading q^* which differs only slightly from the uniform loading q.

The method employed for the solution of this problem is developed in three major stages. The first stage involves the selection of w. Obviously an arbitrary choice of w cannot guarantee a solution such that $q^* \sim q$. Therefore a procedure for the selection of a reasonable w by consideration of contour curves is presented. The second stage is devoted to the determination of a stress function ϕ which, together with w, satisfies the MDV compatibility equation exactly. The third stage of the development is concerned with computation of the ratio q^*/q and comparing the magnitude of this ratio with unity.

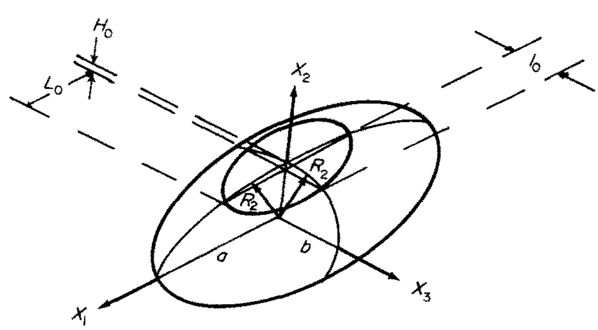


Fig. 1. Geometry of the middle surface of a prolate spheroidal shell showing the shallow spheroid.

2. THE CONTOUR EQUATION

In this section an equation (called the contour equation) in which w is the only dependent variable is separated from the MDV equations under the assumption that the loading is uniform, (i.e. q is constant). Then a transformation from the independent variables of this equation to a new coordinate system (λ, t) is introduced where it is assumed that w is dependent upon λ only.

The MDV-equations are given by[12]

$$\frac{1}{Eh}\nabla^4\phi + D^2 w = 0 \tag{2.1a}$$

and

$$\frac{Eh^3}{12(1-\nu^2)}\nabla^4 w - D^2\phi = q \tag{2.1b}$$

where the operators ∇^2 and D^2 are defined by

$$\nabla^2(\cdots) = \frac{1}{A_1^2}\frac{\partial^2(\cdots)}{\partial\alpha_1^2} + \frac{1}{A_2}\frac{\partial^2(\cdots)}{\partial\alpha_2^2} \tag{2.2a}$$

$$D^2(\cdots) = \frac{1}{A_1R_2}\frac{\partial^2(\cdots)}{\partial\alpha_1^2} + \frac{1}{A_2R_1}\frac{\partial^2(\cdots)}{\partial\alpha_2^2} \tag{2.2b}$$

where A_1 and A_2 are the Lame coefficients and R_1 and R_2 are the radii of curvature of the cap. In terms of the planform dimensions and the rise of the cap, these parameters are (see Fig. 1)

$$A_1 = b = l_0(1+(H_0/l_0)^2)/(2H_0/l_0) \tag{2.3a}$$

$$A_2 = a = L_0(1+(H_0/l_0)^2)/(2H_0/l_0) \tag{2.3b}$$

$$R_1 = b = A_1 \tag{2.3c}$$

$$R_2 = a/k, \tag{2.3d}$$

$$k = l_0/L_0 = b/a. \tag{2.3e}$$

By operating on eqn (2.1a) with D^2 and eqn (2.1b) with $(1/Eh)\nabla^4$, then adding the resulting equations and observing the commutativity of ∇^2 and D^2, it can be concluded that

$$\nabla^8 w + \mu D^4 w = 0 \tag{2.4a}$$

where

$$\mu = \frac{12(1-\nu^2)}{h^2}. \tag{2.4b}$$

This equation will henceforth be called the "contour" equation. The contour equation will now be transformed to a λ, t-system.

The λ, t-system presented in this paper is one in which the constant λ lines are ellipses formed by intersecting the shell with planes parallel to the boundary. The constant t lines are formed by the intersections of the shell's middle surface with planes containing the X_2-axis (see Fig. 2). This particular coordinate system was selected since the constant λ lines represent a generalization of the known contour lines of w for three extreme cases: the shallow spherical cap ($k = 1$), the infinitely long cylindrical cap ($k = 0$), and the flat elliptical plate[13].

Taking into account the fact that the shell is shallow, the λ, t-coordinate system can be defined by the transformation

$$\alpha_1 = \lambda \cos t \tag{2.5a}$$

$$\alpha_2 = \lambda \sin t \tag{2.5b}$$

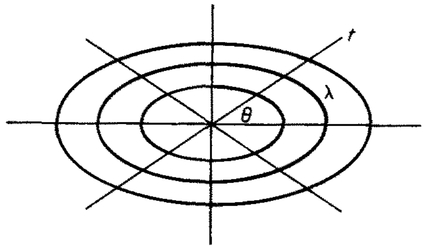


Fig. 2. Geometry of the middle surface of the shallow spheroid showing the λ, t coordinate system.

$$\lambda = \sqrt{1-(X_2/R_1)^2} \tag{2.5c}$$

$$\tan t = k \tan \theta = k(X_3/X_1). \tag{2.5d}$$

Using the chain rule and the assumption that $w = w(\lambda)$, the contour equation transforms to the λ, t-system as

$$\sum_{j=0}^{8}\left(\sum_{m=5}^{8} F_{jm}\bar{\lambda}^m \frac{d^m w}{d\bar{\lambda}^m} + \sum_{m=1}^{4} [F_{jm} + \mu_0 B_{jm}\bar{\lambda}^4]\bar{\lambda}^m \frac{d^m w}{d\bar{\lambda}^m}\right)\cos jt = 0 \tag{2.6}$$

where

$$\mu_0 = \mu b^2 k^4 \lambda_0^4, \qquad \bar{\lambda} = \lambda/\lambda_0 \tag{2.7a,b}$$

$$F_{jm} = H(8,0,j,m) + 4k^2 H(6,2,j,m) + 6k^4 H(4,4,j,m) + 4k^6 H(2,6,j,m) + k^8 H(0,8,j,m)$$

$$B_{jm} = \begin{cases} H(4,0,j,m) + 2H(2,2,j,m) + H(0,4,j,m) \ldots 0 \le j \le 4 \\ 0 \ldots 5 \le j \le 8 \end{cases} \tag{2.8b}$$

and the H-coefficients arise from the transformation of the partial derivatives and are given in Ref. [9]. The constant λ_0 is the value of λ at the boundary.

The ordinary differential eqn (2.6) is the transformed contour equation. Note that in terms of the new coordinate system, the clamped boundary conditions are easily specified by requiring that w and $(dw/d\bar{\lambda})$ vanish at $\bar{\lambda} = 1$.

3. AN APPROXIMATE SOLUTION OF THE CONTOUR EQUATION

In order to be consistent with the assumption that $w = w(\lambda)$ the solution of the transformed contour equation must be independent of the variable t. Therefore, since the cos jt are linearly independent, each coefficient of the cosine terms in (2.6) must vanish. Thus

$$\sum_{m=5}^{8} F_{jm}\bar{\lambda}^m \frac{d^m w}{d\bar{\lambda}^m} + \sum_{m=1}^{4} [F_{jm} + \mu_0 B_{jm}\bar{\lambda}^4]\bar{\lambda}^m \frac{d^m w}{d\bar{\lambda}^m} = 0 \tag{3.1}$$

where the j index refers to the coefficient of cos jt.

The values for the j index are 0, 1, 2, . . . , 8. Thus (3.1) is a set of nine eighth order ordinary differential equations with w as the dependent variable. Evaluation of F_{jm} and B_{jm}, however, reveals that eqn (3.1) is an identity for odd values of the j index. Hence, (3.1) is reduced to a set of five differential equations corresponding to the values $j = 0$, 2, 4, 6 and 8. If an exact solution $w = w(\lambda)$ exists, then each of the five differential eqn (3.1) must be satisfied by this solution. Further, if such an exact solution exists then it can be synthesized from the solutions to each of (3.1) for $j = 0$, 2, 4, 6, 8. Hence the solutions $w^{(j)}$, to each of the differential eqns (3.1) corresponding to the index values $j = 0$, 2, 4, 6 and 8 shall be sought.

It will be assumed that each of the $w^{(j)}$ can be expanded in a Maclaurin series. By doing this, those solutions which are unbounded at $\bar{\lambda} = 0$ are suppressed "a priori". Then

$$w^{(j)} = \sum_{n=0}^{\infty} w_n^{(j)}\bar{\lambda}^n \tag{3.2}$$

where the $w_n^{(j)}$ are constants.

By substituting (3.2) into (3.1) the recursion formulas for $w_n^{(j)}$ can be obtained using standard techniques. From these recursions formulas the solutions $w^{(j)}$ which are of the form (3.2) can be constructed.

It can be shown that[14], [9]

$$w = C_1E(x) + C_2F(x) + C_3 \tag{3.3}$$

where C_1, C_2 and C_3 are arbitrary constants and

$$x = \tfrac{1}{4}\sqrt{\mu_0 s(k)\lambda^2} \tag{3.3a}$$

where $s(k)$ is the shape function

$$s(k) = \frac{H(8,0,0,1)+4H(6,2,0,1)+6H(4,4,0,1)+4H(2,6,0,1)+H(0,8,0,1)}{H(8,0,0,1)+4k^2H(6,2,0,1)+6k^4H(4,4,0,1)+4k^6H(2,6,0,1)+k^8H(0,8,0,1)} \tag{3.4}$$

and

$$E(x) = \sum_{m=0}^{\infty}(-1)^m\frac{x^{2m+1}}{(2m+1)!^2} \tag{3.5a}$$

$$F(x) = \sum_{m=0}^{\infty}(-1)^m\frac{x^{2m}}{(2m)!^2} \tag{3.5b}$$

The measure of (3.3) as an approximate solution to (3.1) is made in section 5 following the determination of ϕ.

4. THE DETERMINATION OF ϕ AND Q^*

Since the dependency of ϕ on $\bar{\lambda}$ only has not been established, the stress function may be expanded as

$$\phi = \delta\Phi = \delta\sum_{n=0}^{\infty}\sum_{j=0}^{n}\Phi_{nj}\xi_1^{\,j}\xi_2^{\,n-j}\ldots n, j = 2, 4, 6, \ldots \tag{4.1a}$$

where the Φ_{nj} are unknown coefficients,

$$\delta = \frac{Eh^2}{\sqrt{12(1-\nu^2)s(k)}} \tag{4.1b}$$

and

$$\mu_1 = \frac{\sqrt[4]{\mu_0 s(k)}}{2\lambda_0} \qquad \begin{aligned}\xi_1 &= \mu_1\alpha_1\\ \xi_2 &= \mu_1\alpha_2.\end{aligned} \tag{4.1c}$$

The requirements that n and j are even follows directly from symmetry considerations. Future developments may be simplified by expanding w as given in (3.3) in the form

$$w = \sum_{n=0}^{\infty}\sum_{j=0}^{n} w_{nj}\xi_1^{\,j}\xi_2^{\,n-j} \qquad n, j = 2, 4, 6, \ldots \tag{4.2}$$

where the w_{nj} are known coefficients which are expressed in terms of the constants C_1, C_2 and C_3. The substitution of (4.2) and (4.1a) into the MDV equations can be facilitated by defining new operators

$$\bar{\nabla}^2 = \frac{\partial^2}{\partial\xi_1^2} + k^2\frac{\partial^2}{\partial\xi_2^2} \qquad \bar{D}^2 = \frac{\partial^2}{\partial\xi_1^2} + \frac{\partial^2}{\partial\xi_2^2}. \tag{4.3}$$

Using these aforementioned operators the MDV equations can be rewritten as

$$\bar{\nabla}^4\Phi = -4\bar{D}^2 w \tag{4.4a}$$

$$\bar{D}^2\Phi = \frac{s(k)}{4}\bar{\nabla}^4 w - Q \tag{4.4b}$$

where

$$Q = \frac{4b^2 q}{Ehk^4}. \tag{4.4c}$$

Substitution of (4.1) and (4.2) into (4.4a, b) yields equations for determining the coefficients Φ_{nj}. An examination of the general form of these resulting equations is useful at this time.

From (4.4b) with $n = 0$ and $j = 0$, we get

$$\Phi_{22} + \Phi_{20} = k^*_{00} C_2 - Q/2 \tag{4.5}$$

where k^*_{00} is a polynomial in k. Inspection of the equations obtained for other values of n and j show that the coefficients Φ_{22} and Φ_{20} appear only in eqn (4.5). Rather than solving for, say Φ_{20}, in terms of the unknowns C_2 and Φ_{22}, it is convenient to introduce a new arbitrary constant, C_4, and a new equation

$$\Phi_{22} - \Phi_{20} = C_4. \tag{4.6}$$

Then both Φ_{22} and Φ_{20} can be solved for in terms of the unknowns C_2 and C_4. In matrix form, the above two equations appear as

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{Bmatrix} \Phi_{22} \\ \Phi_{20} \end{Bmatrix} = \begin{Bmatrix} k^*_{00} \\ 0 \end{Bmatrix} C_2 + \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} C_4 - \begin{Bmatrix} Q/2 \\ 0 \end{Bmatrix}. \tag{4.7}$$

The equations for the remaining Φ_{nj} can be grouped as follows:

$$\begin{bmatrix} 24 & 8k^2 & 24k^4 \\ \hline 0 & 2 & 12 \\ 12 & 2 & 0 \end{bmatrix} \begin{Bmatrix} \Phi_{44} \\ \Phi_{42} \\ \Phi_{40} \end{Bmatrix} = \begin{Bmatrix} -16 \\ \hline k^*_{20} \\ k^*_{22} \end{Bmatrix} C_1 \tag{4.8}$$

$$\begin{bmatrix} 0 & 24 & 48k^2 & 360k^4 \\ 360 & 48k^2 & 24k^4 & 0 \\ \hline 0 & 0 & 2 & 30 \\ 0 & 12 & 12 & 0 \\ 30 & 2 & 0 & 0 \end{bmatrix} \begin{Bmatrix} \Phi_{66} \\ \Phi_{64} \\ \Phi_{62} \\ \Phi_{60} \end{Bmatrix} = \begin{Bmatrix} 16 \\ 16 \\ \hline k^*_{40} \\ k^*_{42} \\ k^*_{44} \end{Bmatrix} C_2 \qquad \text{etc.}\ldots \tag{4.9}$$

In condensed notation, eqns (4.8) and (4.9) and all subsequent equations can be written as

$$[K]_n\{\Phi\}_n = \{k^*\}_n C_N \qquad N = \begin{cases} 1 \text{ when } n = 4, 8, 12, \ldots \\ 2 \text{ when } n = 2, 6, 10, \ldots \end{cases} \tag{4.10}$$

where the elements, k^*_{pq} of the lower partition of $\{k^*\}_n$ are polynomials which are dependent upon k only, and the C_1 and C_2 are the same arbitrary constants that appear in (3.3); the dashed lines in (4.8) and (4.9) partition the matrices is such a way as to separate the MDV compatibility equation (upper partition) from the MDV equilibrium equations (lower partition).

The arbitrary constants C_1, C_2, C_3, and C_4 may be obtained by specifying w, $\mathrm{d}w/\mathrm{d}\bar{\lambda}$, u_1 and u_2 (or T_1 and T_2) on the edge where $\bar{\lambda} = 1$.

From (4.7) it is clear that the specification of C_2 and C_4 implies a unique solution for Φ_{22} and Φ_{20}. Also, it can be shown from (4.8) that the specification of C_1 implies a unique solution for Φ_{44},

Φ_{42}, and Φ_{40} in the case that $0 < k < 1$. The remaining Φ_{nj} are over-determined; which implies that the choice of (3.3) for w and the form (4.1a) for Φ does not lead to an exact solution to the MDV equations in the case that Q is a constant.

At this point, an approximate solution will be sought by rewriting the equilibrium equation (4.4b) in the form

$$Q^* = \frac{s(k)}{4}\bar{\nabla}^4 w - \bar{D}^2\Phi \tag{4.11}$$

where Q^* is now permitted to be a nonuniform loading.

By discarding the required number of rows below the dashed lines in $[K]_n$ along with the corresponding rows in $\{k^*\}_n$ so as to make $[K]_n$ a square matrix, solutions can be obtained for all Φ_{nj}. The resulting Φ, together with the choice of w (3.3), will then satisfy the MDV compatibility eqn (4.4a), but they will not satisfy the MDV equilibrium eqn (4.4b). However, Φ and w can be substituted into the equilibrium eqn (4.11) to yield a nonuniform loading Q^*. If Q^* differs only slightly from a constant Q, then these expressions for Φ and w will be considered as constituting an approximate solution to the uniformly loaded cap.

After determining the Φ_{nj} by the above-described row deletion process, it can be shown that eqn (4.11) can be written

$$Q^* = Q + C_1\psi_1(\xi_1, \xi_2) + C_2\psi_2(\xi_1, \xi_2) \tag{4.12a}$$

where

$$\psi_1 = \sum_{n=2,6,10,\ldots}^{\infty} \sum_{j=0,2,4,\ldots}^{n} Q^*_{nj}\xi_1{}^j\xi_2{}^{n-j} \tag{4.12b}$$

and

$$\psi_2 = \sum_{n=4,8,12,\ldots}^{\infty} \sum_{j=0,2,4,\ldots}^{n} Q^*_{nj}\xi_1{}^j\xi_2{}^{n-j} \tag{4.12c}$$

and the Q^*_{nj} are constants which are determined from eqns (4.10) after the row-deletion process has been imposed.

5. THE EVALUATION OF Q^*/Q

In the previous section an exact solution was obtained for a clamped shallow spheroidal shell subject to a nonuniform loading Q^*. The error associated with using this solution as an approximate solution to the uniformly loaded cap will be measured by the deviation of the ratio Q^*/Q from unity.

So far, only the conditions that w and $\mathrm{d}w$ vanish at $\bar{\lambda} = 1$ have been considered, leaving the remaining two boundary conditions which concern either the inplane displacements, u_1 and u_2, or the in-plane stress resultants T_1 and T_2, unspecified. It will now be shown that $Q^* \sim Q$ without embracing the lengthy algebraic manipulations involved in explicitly specifying and satisfying the remaining two boundary conditions. First, introduce a nondimensional displacement

$$\bar{w} = w/w_0 \tag{5.1a}$$

where

$$w_0 = w(\bar{\lambda} = 0) \tag{5.1b}$$

is the displacement at the center of the cap, and redefine the constants of integration as

$$c_1 = C_1/w_0 \qquad c_2 = C_2/w_0 \qquad c_3 = C_3/w_0. \tag{5.1c}$$

Then by imposing the conditions that $\bar{w}$ and $d\bar{w}$ vanish at $\bar{\lambda} = 1$, and $\bar{w} = 1$ at $\bar{\lambda} = 0$, the constants c_1, c_2 and c_3 may be evaluated. Thus, $\bar{w}$ is completely determined, and w is determined to within the value of w_0.

Now, let w_0^P be the maximum normal displacement of a clamped elliptical plate with the same planform, thickness and material properties as those of the shallow spheroid, and loaded uniformly by Q.

From plate theory [13],

$$w_0^P = pQ \tag{5.2a}$$

where

$$p = \frac{\sqrt{12(1-\nu^2)}}{32(3+2k^2+3k^4)}[k^2 l_0 \lambda_0 / h]^2. \tag{5.2b}$$

Then, substituting (5.1) and (5.2) into (4.12), yields

$$Q^*/Q - 1 = \left(\frac{w_0}{w_0^P}\right)\psi \tag{5.3a}$$

where

$$\psi = pc_1\psi_1 + pc_2\psi_2. \tag{5.3b}$$

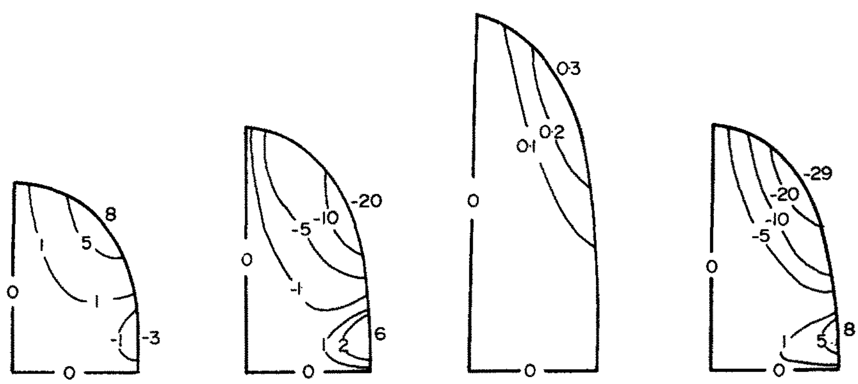
Thus, the deviation of Q^*/Q from unity is given in terms of the known error function ψ multiplied by the unknown ratio of central displacements w_0/w_0^P.

A bound on the error associated with using w from (3.3) and Φ from (4.1a) as the solution can be found without computing w_0 explicitly by noting that whenever

$$|w_0/w_0^P| < 1$$

the maximum error at any point is less than the known function ψ.

The function ψ has been computed for four representative shallow spheroids and the results are plotted in Fig. 3. From the curves of constant ψ plotted in Fig. 3, it is seen that for a given geometry, the largest errors always occur at isolated points on the boundary, which points are removed from the major and minor axes. This latter feature is the direct consequence of a



$k =$	0·65	0·50	0·35	0·50
$\frac{H_0}{2l_0} =$	0·10	0·10	0·10	0·05
$\frac{H_0}{h} =$	1·33	1·33	1·33	2·00

Fig. 3. Contours of 100ψ for quadrants of various shells.

decision to delete certain rows in eqn (4.10) so as to assure an exact solution on the major and minor axes. A different choice for the row deletion process could obviously have been made, and would have resulted in a modified form of the error function ψ.

As expected, the geometrical effect was such as to give the most accurate results as the three extreme geometries were approached i.e., circular cap, infinitely long cylindrical cap, and flat elliptical plate. However, even for the worst case considered, $k = 0{\cdot}5$, $H_0/h = 2{\cdot}00$ and $H_0/2l_0 = 0{\cdot}05$, the maximum error, which occured at the point on the boundary corresponding to $\theta = 60°$, was 29%. For this case the error is still less than 10% over most of the interior of the shell. Note again that these values represent an upper bound to the error.

6. SUMMARY AND CONCLUSIONS

The analysis of a clamped shallow spheroidal shell under uniform pressure was made using a semi-inverse contour method. As a means for arriving at a suitable choice for w, the MDV equations were reduced to one equation with w as the dependent variable. Then making the assumption that $w = w(\lambda)$, a functional form for $w(\lambda)$ was chosen which is a generalization of the solutions for the clamped spherical cap and the clamped elliptical plate. A solution for ϕ was then obtained which, together with the chosen w, satisfied the compatibility equation exactly, and which satisfied the equilibrium equation exactly in the case that the surface loading is non-uniform. The maximum differences between this nonuniform loading and the given uniform load were shown to be confined to isolated points on the boundary. It is reasonable to conclude that by confining the errors in the loading to zones near the boundary, the effect of these errors on the overall behavior of the shell is reduced.

The numerical calculations also demonstrate that the solutions more accurately predict the behavior of the uniformly loaded spheroid as $k \to 1$, or as $k \to 0$, or as $H_0 \to 0$, or as the thickness increases.

In both this solution and the solution obtained independently by Jones and Mazumdar[11] the assumption is made that $w = w(\lambda)$. It is demonstrated herein that such an assumption cannot lead to an exact solution for the uniformly loaded shell. In addition, it is further assumed in Ref. [11] that $\phi = \phi(\lambda)$; no such restriction is made on the form of the stress function in this paper. Further, an upper bound on the error associated with this solution has been presented. No measure of the accuracy of the solution in Ref. [11] is available.

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